UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE : **MATH1202**

ASSESSMENT PATTERN : **MATH1202B**

MODULE NAME : **Algebra 2**

DATE : **18-May-11**

TIME : **14:30**

TIME ALLOWED : **2 Hours 0 Minutes**

2010/11-MATH1202B-001-EXAM-199

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*

The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Give the definition of a *group*, defining the terms you use.

(b) Let $\mathbb{R}_+$ denote the set of positive reals, and let G be the set of all functions $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$. Determine whether or not G forms a group under the given operation $\star$, justifying your answer:

(i) $(f \star g)(x) = f(x)g(x)$;

(ii) $(f \star g)(x) = f(g(x))$;

(iii) $(f \star g)(x) = |f(x) - g(x)|$.

2. (a) Let G be a finite group and H a subgroup. Prove that $|H|$ divides $|G|$.

(b) Prove that if H and K are subgroups of a group G then $H \cap K$ is also a subgroup of G. Let G be a group of order 35 and let H and K be subgroups of order 5 and 7 respectively. Prove that $H \cap K = \{e\}$ and deduce that every element of G can be written uniquely in the form hk for some $h \in H, k \in K$.

3. (a) Let A be an $n \times n$ matrix. Give the definition of $\det(A)$.

(b) Prove that $\det A = \det A^T$.

(c) Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^3$, and define the 3×3 real matrix B by $b_{ij} = \mathbf{v}_i.\mathbf{v}_j$. Using (b), prove that $\det B \geq 0$.

(d) Evaluate the determinant of

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 3 & 2 & 1 \\ 2 & 7 & 5 & 2 \\ 1 & 0 & 1 & 1 \end{pmatrix}.$$

4. Let $A = \begin{pmatrix} 2 & 3 \\ 4 & -2 \end{pmatrix}$.

(i) Find an invertible matrix P such that $P^{-1}AP$ is diagonal.

(ii) Find A^n (for positive integers n).

(iii) Find four (complex) solutions to $X^2 = A$. Show that there are no other solutions.

5. (a) Let A be an $n \times n$ matrix over $\mathbb{C}$ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_r$. Give the definition of:

(i) the *eigenspace* E_{λ_i} associated to λ_i;

(ii) the *geometric multiplicity* e_i of A;

(iii) the *characteristic polynomial* $c_A(t)$ of A;

(iv) the *algebraic multiplicity* f_i of A.

(b) Prove that if A has n distinct eigenvalues then A is diagonalisable.

(c) State (without proof) a necessary and sufficient condition in terms of the e_i and f_i for A to be diagonalisable. Determine for which values of a, b the matrix

$$A = \begin{pmatrix} 1-b & b \\ 1-a-b & a+b \end{pmatrix}$$

is diagonalisable, justifying your answer.

6. (a) Let A be a real symmetric matrix. Prove that all the eigenvalues of A are real.

(b) Orthogonally diagonalise $A = \begin{pmatrix} 5 & -\sqrt{3} \\ -\sqrt{3} & 7 \end{pmatrix}$.

(c) Using part (b), sketch the curve $5x^2 - 2\sqrt{3}xy + 7y^2 = 4$, explaining your answer.